MTH 295
Fall 2019
Homework 2
Due Thursday, 9/19 Name: Key

1) Solve the IVP $\frac{dy}{dx} = -y(x^2+1)$, $y(-1)=1$.

The equation is separable -

$$\frac{dy}{y} = -(x^2+1)\,dx$$

$$\ln|y| = -\frac{x^3}{3} - x + c$$

$$|y| = e^{-\frac{x^3}{3} - x + c}$$

$$|y| = Ce^{-\frac{x^3}{3} - x}$$

apply condition, $y=1$ when $x=-1$

$$1 = Ce^{+\frac{1}{3}+1}$$

$$= Ce^{4/3}$$

$$C = e^{-4/3}$$

So $|y| = e^{-\frac{x^3}{3} - x - \frac{4}{3}}$

There appear to be 2 solutions but

if $y>0$ then $|y| = y = e^{-\frac{x^3}{3} - x - \frac{4}{3}}$

which satisfies the condition because

$$1 = e^{\frac{1}{3}+1-4/3} = e^0 = 1$$

but if $y<0$ then $y = -e^{-\frac{x^3}{3} - x - \frac{4}{3}}$

which does not satisfy the condition

because $1 \neq -e^{\frac{1}{3}+1-4/3} = -1$

So there is a unique solution -

$$\boxed{y = e^{-\frac{x^3}{3} - x - \frac{4}{3}}}$$

2) Solve the IVP $\dfrac{dy}{dx}=\dfrac{y\cos x}{1+2y^2}$, $y(0)=1$ (Note: your solution will be implicit)

$$\left(\frac{1+2y^2}{y}\right)dy = \cos x\,dx$$

$$\int\left\{\frac{1}{y}+2y\right\}dy = \int\cos x\,dx$$

$$\ln|y| + y^2 = \sin x + C$$

if $y(0)=1$ then

$$1 = C$$

$$\ln|y|+y^2 = \sin x + 1$$

$$e^{\ln|y|+y^2} = e^{\sin x+1}$$

$$|y|e^{y^2} = e^{\sin x+1}$$

Again, it appears there are 2 solutions but

if $y>0$, then $ye^{y^2}=e^{\sin x+1}$

and, applying condition -

$$(1)e^1 = e^1$$

but if $y<0$ then

$(-1)e^1 \neq e^1$ and it must be true that

$$\boxed{ye^{y^2} = e^{\sin x+1}}$$

3) In class we solved the IVP $\dot{x} = kx$, $x(0) = x_0$ and found that the solutions are always exponential. One situation where this is unrealistic is population growth. Populations don't grow without bound. They are constrained by crowding, food supply, etc. A simple model that captures this limited growth is the IVP $\dot{x} = kx\left(1 - \frac{x}{N}\right)$, $x(0) = x_0$ where $x = x(t)$ is the population at time t, $k > 0$ is the growth constant, and N is called the carrying capacity of the environment.

We can sometimes say a lot about behavior without actually solving an equation. In fact, it is frequently impossible to find a solution but possible to at least determine behavior. Without solving the IVP, show the following.

a) If $x = 0$ then $\dot{x} = 0$ for all t and thus $x = 0$ for all t.

If $x = 0$ then $\dot{x} = k(0)(1-0) = 0$ and x = constant $= 0$ for all t.

b) If $x = N$ then $\dot{x} = 0$ for all t and thus $x = N$ for all t.

If $x = N$ then $\dot{x} = kN(1 - \frac{N}{N}) = 0$ and x = constant $= N$ for all t.

c) If $x << N$ then the problem reduces to the simpler problem of exponential growth that we did in class and thus the growth should be exponential for at least small t.

If $x << N$ then $\dot{x} = kx(1 - \frac{x}{N}) \approx kx > 0$ so we have growth.

(we solved this equation in class)

d) If $0 < x < N$ there will be growth.

If $0 < x < N$ then $\frac{x}{N} < 1$, $(1 - \frac{x}{N}) > 0$ and

$\dot{x} = kx(1 - \frac{x}{N}) > 0$ so there is growth.

e) If $x > N$ there will be decay.

If $x > N$ then $\frac{x}{N} > 1$, $(1 - \frac{x}{N}) < 0$

and $\dot{x} = kx(1 - \frac{x}{N}) < 0$ so there is decay.

f) Solve the initial value problem. Notice the equation is separable. You can write the solution $x(t)$ as an explicit function of t.

$$\frac{dx}{dt}=kx\left(1-\frac{x}{N}\right)$$

$$\frac{dx}{x\left(1-\frac{x}{N}\right)}=k\,dt$$

$$\frac{N\,dx}{x(N-x)}=k\,dt$$

$$N\left\{\frac{1}{Nx}+\frac{1}{N(N-x)}\right\}dx=k\,dt$$

$$\int\left\{\frac{1}{x}+\frac{1}{N-x}\right\}dx=k\,dt$$

$$\ln\left|\frac{x}{N-x}\right|=kt+c$$

$$\left|\frac{x}{N-x}\right|=ce^{kt}$$

$$\frac{x}{|N-x|}=ce^{kt}\text{, since } x\geq 0$$

$$\frac{1}{x(N-x)}=\frac{A}{x}+\frac{B}{N-x}$$

$$1=AN-Ax+Bx$$

so $A=\frac{1}{N}$, $B=A=\frac{1}{N}$

There appear to be 2 cases –

Case 1

if $N>x$ then

$$\frac{x}{N-x}=ce^{kt}$$

if $x(0)=x_0$ then

$$\frac{x_0}{N-x_0}=c$$

and $\frac{x}{N-x}=\frac{x_0}{N-x_0}e^{kt}$

$$x(N-x_0)=(N-x)x_0e^{kt}$$

$$x=\frac{Nx_0e^{kt}}{N-x_0+x_0e^{kt}}$$

Case 2

if $x>N$ then

$$\frac{x}{x-N}=ce^{kt}$$

$$\frac{x_0}{x_0-N}=c$$

$$\frac{x}{x-N}=\frac{x_0}{x_0-N}e^{kt}$$

$$x(x_0-N)=(x-N)x_0e^{kt}$$

$$x=\frac{-Nx_0e^{kt}}{x_0-N-x_0e^{kt}}$$

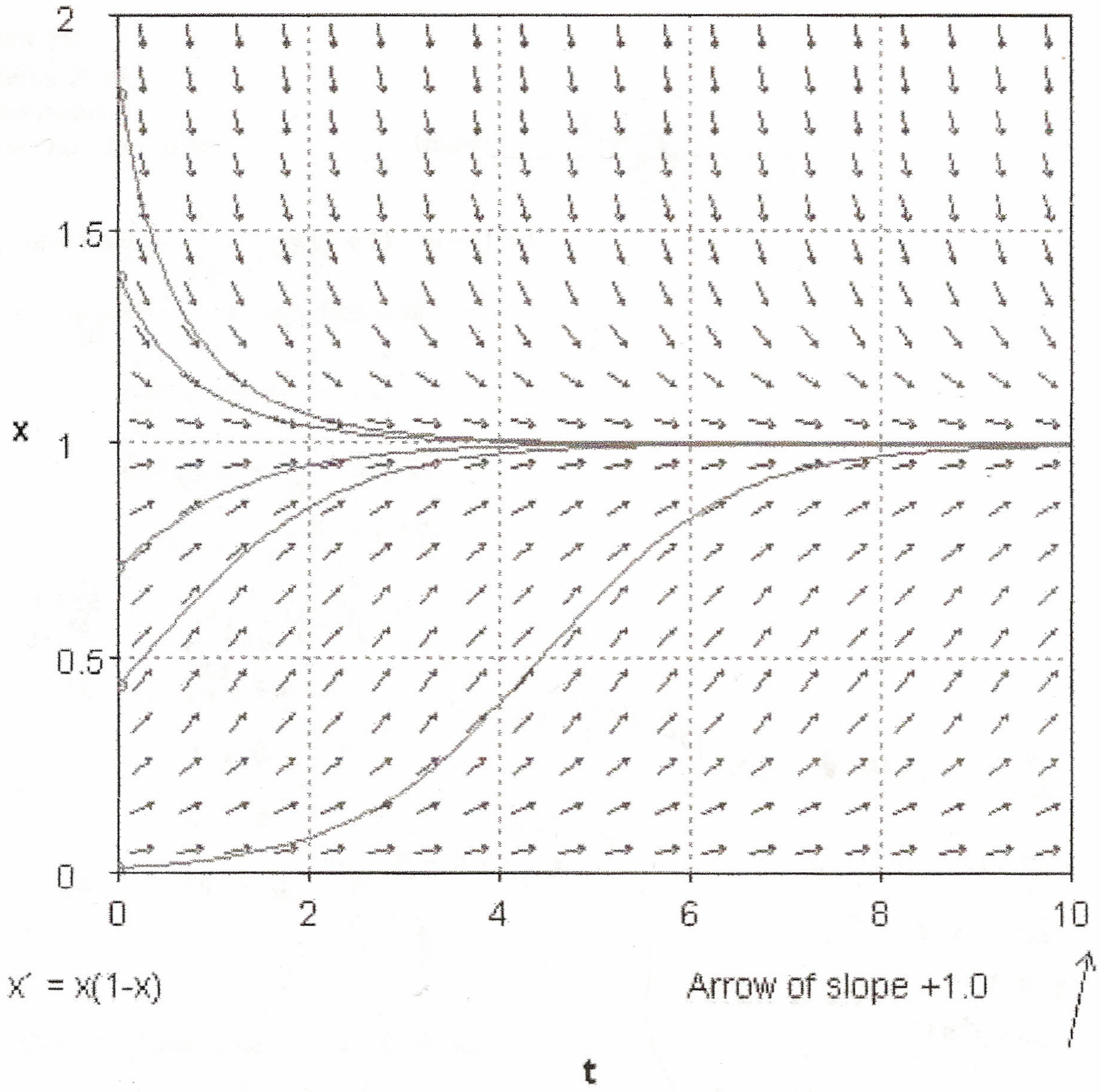
$$=\frac{Nx_0e^{kt}}{N-x_0+x_0e^{kt}}$$

which is the same!!

so $$\boxed{x=\frac{Nx_0e^{kt}}{N-x_0+x_0e^{kt}}}$$

As we have seen, although there appear to be multiple solutions there is actually one unique solution to the IVP.

g) Using d-field, construct the direction field of $x(t)$ vs t for $x,t \geq 0$ and include a few solutions to verify all of the behavior that you found on the previous page. In particular, include large enough t to capture the behavior. You should find that all solutions eventually approach $x = N$ for sufficiently large t (unless, of course, $x_0 = 0$). Thus the name "carrying capacity". You'll have to choose some different values for the parameters k and N so go ahead and play around. Your choices should not change the qualitative behavior.



I let $N = 1$, $k = 1$.

Note that $x \to N$ for all x_0.

If $x_0 > N$ then $\frac{dx}{dt} < 0$.

If $x_0 < N$ then $\frac{dx}{dt} > 0$.

Also note if $x_0 \ll N$ then growth is exponential until x increases. In the graph above the lowest curve looks exponential until about $t = 4$ where there is a point of inflection.